# On Lacunary Polynomial Perfect Powers

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# Polynomial Representations

A polynomial which is easy to write:

$$f(x) = x^{123705}y^{9432} + 9\,x^{95911}y^{7510} + 3\,x^{82470}y^{6288} + 27\,x^{68117}y^{5588} + 18\,x^{54676}y^{4366}$$
$$+ 3\,x^{41235}y^{3144} + 27\,x^{40323}y^{3666} + 27\,x^{26882}y^{2444} + 9\,x^{13441}y^{1222} + 1$$

How to represent this polynomial?

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## Dense Representation

For a degree-$n$ polynomial in $k$ variables:

- Store a $k$-dimensional array of every possible coefficient
- Size $\approx n^k$
- This example: more than 1 GB!

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## Sparse Representation

For a degree-$n$ polynomial in $k$ variables with $t$ terms:

- Store a list of $t$ coefficient-exponent tuples
- Size $\approx kt \log_2 n$
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- Store a list of $t$ coefficient-exponent tuples
- Size $\approx kt \log_2 n$
- This example: less than 1 KB
- More natural representation
- Default in most CAS
- Can be exponentially more compact

# Computing with Polynomials

Lots of things are easy when polynomials are dense:

- GCD, factorization, Euclidean division
- Relative Primality
- Square-freeness
- Divisibility
- Reducibility
- Evaluation, Differentiation
- Interpolation
- Root finding
- Low-degree factors
- Jacobi symbols, perfect square detection

# Computing with Polynomials

Some things are hard when polynomials are sparse:

- GCD, factorization, Euclidean division
- Relative Primality (Plaisted '84)
- Square-freeness (Karpinski & Shparlinski '99)
- Divisibility
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  (Ben-Or & Tiwari '88; Kaltofen & Lee '03; G. & R. '07)
- Root finding (Cucker, Koiran, Smale '99)
- Low-degree factors (Lenstra '99; Kaltofen & Koiran '05,'06)
- Jacobi symbols, perfect square detection (Shparlinski '00)

# Computing with Polynomials

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- Jacobi symbols, perfect square detection (Shparlinski '00)
- Perfect power detection Today!

# Back to the example

$$f(x) = x^{123705}y^{9432} + 9\,x^{95911}y^{7510} + 3\,x^{82470}y^{6288} + 27\,x^{68117}y^{5588} + 18\,x^{54676}y^{4366}$$
$$+ 3\,x^{41235}y^{3144} + 27\,x^{40323}y^{3666} + 27\,x^{26882}y^{2444} + 9\,x^{13441}y^{1222} + 1$$
$$= \left(x^{41235}y^{3144} + 3\,x^{13441}y^{1222} + 1\right)^3$$
$$r = 3$$
$$h(x) = x^{41235}y^{3144} + 3\,x^{13441}y^{1222} + 1$$

- We will use this notation consistently.

# Problems to Solve

1. Given $f$, determine whether $f = h^r$ for any $h$ and $r \geq 2$.
2. If so, find one such $r$.
3. Given $r$, find one such $h$.

Algorithm Requirements:

- Cost polynomial in $k$, $t$, and $\log n$
- When R $= \mathbb{Z}$, also polynomial in $\log \|f\|_\infty$

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Algorithm Requirements:

- Cost polynomial in $k$, $t$, and $\log n$
- When R $= \mathbb{Z}$, also polynomial in $\log \|f\|_\infty$

First, we solve (1) and (2) for univariate integer polynomials.

# Perfect Power Detection Algorithm

**Input:** $f \in \mathbb{Z}[x]$

**Output:** $r \geq 2$ s.t. $f = h^r$ for some $h$, or FALSE

**for** each possible $r$ **do**

Choose random $\alpha \in \mathbb{Z}$

**if** $f(\alpha)$ is a perfect $r^{\text{th}}$ power **then**

**return** $r$

**end do**

**return** FALSE

Problem: Can't evaluate over $\mathbb{Z}$

# Perfect Power Detection Algorithm

**Input:** $f \in \mathbb{Z}[x]$

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**for** each possible $r$ **do**

Probabilistically choose $p$ with $p \nmid \operatorname{disc}(f)$

Choose random $\alpha \in \mathbb{F}_p$

**if** $f(\alpha)$ is a perfect $r^{\text{th}}$ power **then**

**return** $r$

**end do**

**return** FALSE

Problem: Can't determine $r^{\text{th}}$-poweredness over $\mathbb{F}_p$.

# Perfect Power Detection Algorithm

**Input:** $f \in \mathbb{Z}[x]$

**Output:** $r \geq 2$ s.t. $f = h^r$ for some $h$, or FALSE

**for** each possible prime $r$ **do**

Probabilistically choose $p$ with $p \nmid \mathrm{disc}(f)$

$q \leftarrow p^{r-1}$

Choose random $\alpha \in \mathbb{F}_q$

**if** $f(\alpha)^{(q-1)/r} = 1$ **then**

**return** $r$

**end do**

**return** FALSE

Problem: Need some deeper math

# Perfect Power Detection Algorithm

**Input:** $f \in \mathbb{Z}[x]$

**Output:** $r \geq 2$ s.t. $f = h^r$ for some $h$, or FALSE

**for** each prime $r < 2\log_2 \|f\|_1$ **do**

Probabilistically choose $p$ with $p \nmid \mathrm{disc}(f)$

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Choose random $\alpha \in \mathbb{F}_q$

**if** $f(\alpha)^{(q-1)/r} = 1$ **then**

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Problem: Need some deeper math

1. Can restrict to $r < 2\log_2 \|f\|_1$

# Perfect Power Detection Algorithm

**Input:** $f \in \mathbb{Z}[x]$

**Output:** $r \geq 2$ s.t. $f = h^r$ for some $h$, or FALSE

**for** each prime $r < 2\log_2 \|f\|_1$ **do**

Probabilistically choose $p$ with $p \nmid \operatorname{disc}(f)$

$q \leftarrow p^{r-1}$

Choose random $\alpha_1, \alpha_2, \ldots, \alpha_5 \in \mathbb{F}_q$

**if** $f(\alpha_1)^{(q-1)/r} = \cdots = f(\alpha_5)^{(q-1)/r} = 1$ **then**

**return** $r$

**end do**

**return** FALSE

Problem: Need some deeper math

1. Can restrict to $r < 2\log_2 \|f\|_1$
2. Five evaluations guarantees high probability of success

# Bound on $r$

## Theorem

*If $f \in \mathbb{Z}[x]$ is a perfect $r^{\text{th}}$ power and has at least 2 terms, then*

$$r \leq 2 \log_2 \|f\|_1.$$

- Proof arises from orthogonality of DFT matrix
- If $f$ has only one term, the problem is trivial (primality testing).

# Perfect Power Evaluation Witnesses

## Theorem

*Suppose* $f \in \mathbb{F}_q[x]$ *has degree* $n$ *and is not a perfect* $r^{\text{th}}$ *power. Then, if* $q \geq 4n^2$,

$$\#\left\{\alpha \in \mathbb{F}_q : f(\alpha) \text{ is an } r^{\text{th}} \text{ power}\right\} \leq \frac{3q}{4}.$$

- Means that at least $1/4$ of elements will be “good witnesses”.
- Proof uses method of completing the character sum, relying on a theorem of Weil (1948).
- No information on the *distribution* of good witnesses.

# Perfect Power Detection

**Theorem**

*For $f \in \mathbb{Z}[x]$ with degree $n$ and $t$ terms, we can determine whether $f$ is a perfect power using $O(t \log^2 \|f\|_\infty \log^2 n)$ bit operations.*

- Probabilistic Monte Carlo algorithm
  (always fast; correct with arbitrarily high probability)
- For $f \in \mathbb{Q}[x]$, can reduce to $\mathbb{Z}[x]$ by Gauß's Lemma
- The case of $f \in \mathbb{F}_q[x]$ is already handled!
- For $f \in \mathrm{R}[x_1, x_2, \ldots, x_k]$, substitute random values for $x_2, \ldots, x_k$ and work over $\mathrm{R}[x_1]$.

# Previously-Known Methods to Detect Perfect Powers

## Square-Free Decomposition (Yun '76)

Computes $f = g_1^{d_1} g_2^{d_2} \cdots g_k^{d_k}$,
for square-free, relatively prime, nonconstant $g_1, \ldots, g_k$.
$f$ is a perfect power iff $\gcd(d_1, \ldots, d_k) > 1$.

## Newton Iteration

For each $r$, computes a power series $r^{\text{th}}$ root of $f$.
Then Monte Carlo check by random evaluation.

- Follows best method for computing roots of integers
  (Bach & Sorenson '93; Bernstein '98)

# Previously-Known Methods to Detect Perfect Powers

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$f$ is a perfect power iff $\gcd(d_1, \ldots, d_k) > 1$.
Implemented natively in NTL

## Newton Iteration

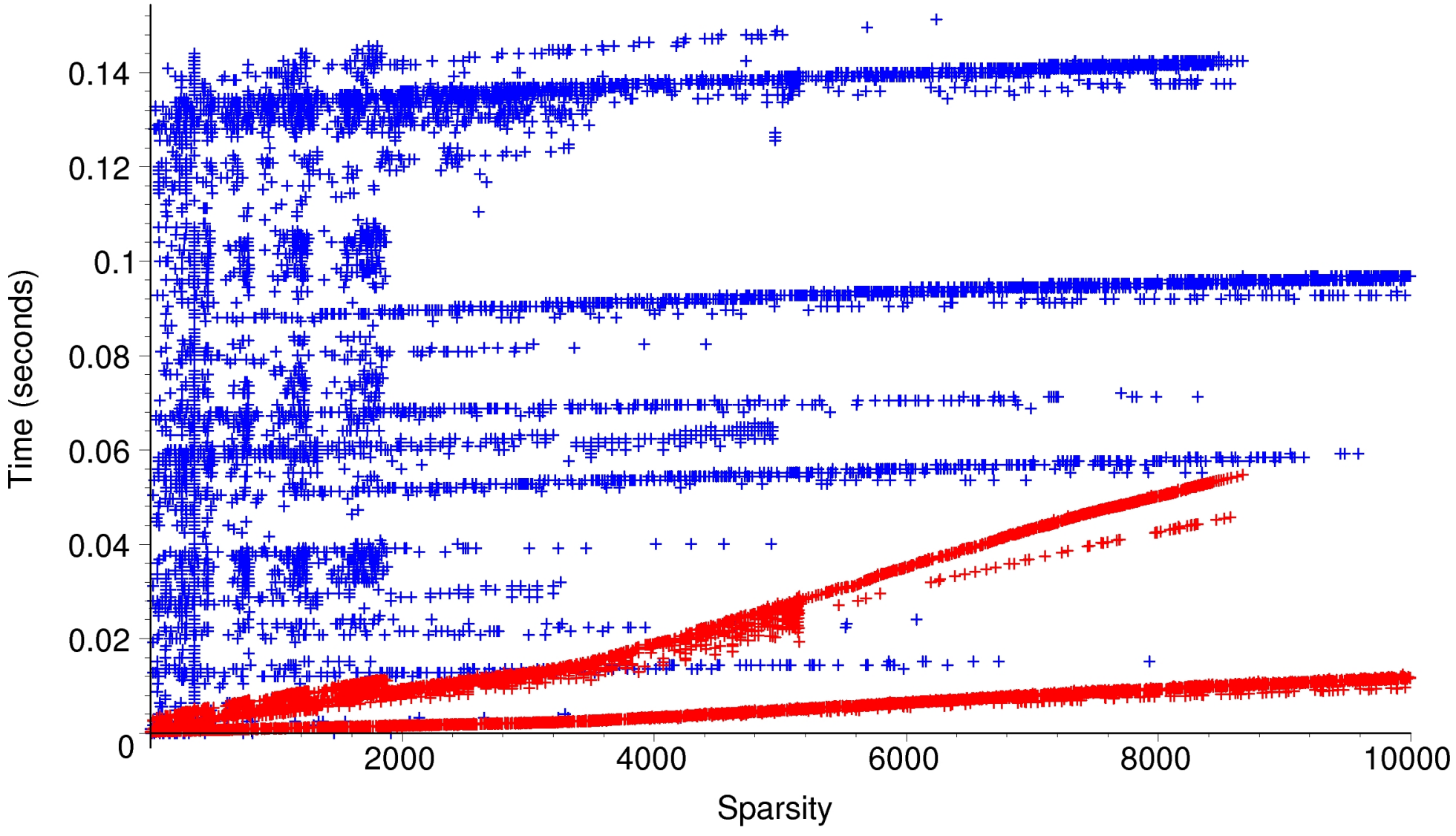
For each $r$, computes a power series $r^{\text{th}}$ root of $f$.
Then Monte Carlo check by random evaluation.
Implemented in NTL by us

- Follows best method for computing roots of integers
  (Bach & Sorenson '93; Bernstein '98)

We also implemented our detection algorithm in NTL.

# Notes on the Timings

- Square-free decomposition was always much slower; not shown
- Millions of trials, ZERO failures
- Our algorithm wins even for dense polynomials (it's really a black-box algorithm)
- Not a fair comparison because we're not computing the root (yet. . . )


Newton iteration (blue) vs. Lacunary Alg. (red) for $n = 10,000$
Time (seconds)
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
2000
4000
6000
8000
10000
Sparsity

# Computing Perfect $r^{\text{th}}$ roots

## Problem

We can determine if $f = h^r$ for some $h$, and find $r$, but how do we compute $h$ given $r$ and (lacunary) $f$?

How dense can the root be?

- This is a very old, well-studied problem
  Erdös ('47); Coppersmith & Davenport ('91); Abbot ('02); Schinzel ('87); Zannier ('07)
- Still not known whether a sparse poly. may have a dense root.
  (Our first thm. proves this for fixed-height integer polynomials)
- We seek only output-sensitive algorithms
  (i.e. assume $h$ is sparse)

# Newton Iteration to Compute Roots

**Input:** $f \in \mathrm{R}[x], r \in \mathbb{N}$ s.t. $f$ is a perfect $r^{\text{th}}$ power
**Output:** $h \in \mathrm{R}[x]$ s.t. $f = h^r$

- $k \leftarrow 1; h \leftarrow \sqrt[r]{f(0)}$
- **while** $k \leq (\deg f)/r$
  - $h \leftarrow h + \dfrac{f - h^r}{r\,h^{r-1}} \mod x^{2k}$
  - $k \leftarrow 2k$
- **end while**

Let's rearrange to give more insight into the computation

# Newton Iteration to Compute Roots

**Input:** $f \in \mathrm{R}[x], r \in \mathbb{N}$ s.t. $f$ is a perfect $r^{\text{th}}$ power
**Output:** $h \in \mathrm{R}[x]$ s.t. $f = h^r$

- $k \leftarrow 1; h \leftarrow \sqrt[r]{f(0)}$
- **while** $k \leq (\deg f)/r$
  - $a \leftarrow \dfrac{f - h^r \bmod x^{2k}}{r x^k}$
  - $h \leftarrow h + \left(\dfrac{a}{h^{r-1}} \bmod x^k\right) \cdot x^k$
  - $k \leftarrow 2k$
- **end while**

Problem: $h^r \bmod x^{2k}$ could be dense

# Sparse Newton Iteration to Compute Roots

**Input:** $f \in \mathrm{R}[x], r \in \mathbb{N}$ s.t. $f$ is a perfect $r^{\text{th}}$ power
**Output:** $h \in \mathrm{R}[x]$ s.t. $f = h^r$

- $k \leftarrow 1; h \leftarrow \sqrt[r]{f(0)}$
- **while** $k \leq (\deg f)/r$
  - $a \leftarrow \dfrac{fh - h^{r+1} \bmod x^{2k}}{rx^k}$
  - $h \leftarrow h + \left(\dfrac{a}{f} \bmod x^k\right) \cdot x^k$
  - $k \leftarrow 2k$
- **end while**

But we can prove $h^{r+1} \bmod x^{2k}$ is *not* dense (assuming the output is sparse).

# Complexity of Computing the Root

Need a conjecture to prove output-sensitive polynomial time:

## Conjecture

Given $h \in \mathrm{R}[x]$ and $r, k \in \mathbb{N}$, we can compute $h^r \bmod x^k$ in time polynomial in:

- The lacunary size of $h$
- The lacunary size of $h^r \bmod x^k$, and
- $\log r$

- Repeated squaring and truncating seems to satisfy this.

## Update

We now have a (different) provable method of computing $r^{\text{th}}$ roots in output-sensitive polynomial time
(almost certainly less efficient in practice).

# Conclusions and Future Directions

## Contributions

- Monte Carlo polynomial-time algorithm to determine if a lacunary polynomial is a perfect power
- Sparsity-sensitive Newton iteration to compute the root (subject to conjectures)

## Future Directions

**Other Models** Straight-line programs, black boxes, . . .

**Other Domains** Sparse integers, approximate, . . .

**Other Problems** Divisibility, reducibility, factorization, . . .

# Some Open Problems

- Given $f \in \mathbb{Z}[x]$ with $t$ nonzero terms, what is the *least* number of terms in $f^2$? $f^r$? $g \circ f$?
- Given a black box (or SLP) for $f \in \mathbb{F}_q[x]$, construct a black box (or SLP) for $f^2 \bmod x^k$.
- Find polynomial-time algorithms for lacunary polynomial divisibility or reducibility tests, or prove they are NP-hard.
- Solve any of these problems for sparse integers.